2413-balloon permutations and the growth of the Möbius function

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Abstract

We show that the growth of the principal Möbius function on the permutation poset is exponential. This improves on previous work, which has shown that the growth is at least polynomial.

We define a method of constructing a permutation from a smaller permutation which we call "ballooning". We show that if β is a 2413-balloon, and π is the 2413-balloon of β, then $\mu[1, \pi] = 2\mu[1, \beta]$. This allows us to construct a sequence of permutations $\pi_1, \pi_2, \pi_3 \dots$ with lengths $n, n+4, n+8, \dots$ such that $\mu[1, \pi_{i+1}] = 2\mu[1, \pi_i]$, and this gives us exponential growth. Further, our construction method gives permutations that lie within a hereditary class with finitely many simple permutations.

We also find an expression for the value of $\mu[1, \pi]$, where π is a 2413-balloon, with no restriction on the permutation being ballooned.

1 Introduction

Let σ and π be permutations of natural numbers, written in one-line notation, with $\sigma = \sigma_1\sigma_2 \dots \sigma_m$, and $\pi = \pi_1\pi_2 \dots \pi_n$. We say that σ is *contained* in π if there is a sequence $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that for any $r, s \in \{1, \dots, m\}$, $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$. We say that π *avoids* σ if π does not contain σ. The set of all permutations is a poset under the partial order given by containment.

A closed interval $[\sigma, \pi]$ in a poset is the sub-poset $\{\tau : \sigma \leq \tau \leq \pi\}$, and a half-open interval $[\sigma, \pi)$ is the sub-poset $\{\tau : \sigma \leq \tau < \pi\}$. The Möbius function $\mu[\sigma, \pi]$ is defined on an interval of a poset as follows: for $\sigma \not\leq \pi$, $\mu[\sigma, \pi] = 0$; for

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all π, $\mu[\pi, \pi] = 1$; and for $\sigma < \pi$,

$$\mu[\sigma, \pi] = -\sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda]$$

In this paper we are principally concerned with the growth of the *principal Möbius function*, $\mu[\pi] = \mu[1, \pi]$.

Applying the Möbius function to the permutation poset was first mentioned by Wilf [7]. Burstein, Jelínek, Jelínková and Steingrímsson [3] ask whether the principal Möbius function is unbounded, which is the first reference to the growth of the Möbius function in the literature. They show that $\mu[\pi] \in \{0, \pm 1\}$, and thus is bounded, if π is a separable permutation, and so is in a hereditary class with simples $\{1, 12, 21\}$. They ask (Question 27) for which classes is $\mu[\pi]$ bounded?

Smith [5] found an explicit formula for the principal Möbius function for all permutations with a single descent. This shows that the growth of the Möbius function is at least quadratic. Jelínek, Kantor, Kynčl and Tancer [4] show how to construct a sequence of permutations where the absolute value of the Möbius function grows according to the seventh power of the length. In the other direction, Brignall, Jelínek, Kynčl and Marchant [1] show that the proportion of permutations of length n with principal Möbius function equal to zero is asymptotically bounded below by $(1 - 1/e)^2 \geq 0.3995$.

We show that, given some permutation β, we can construct a permutation that we call the "2413-balloon" of β. This permutation will have four more points than β. We then show that if π is a 2413-balloon of β, and β is itself a 2413-balloon, then $\mu[\pi] = 2\mu[\beta]$. From this we deduce that the growth of the principal Möbius function is exponential. If $\beta = 25314$ (which is a 2413-balloon), then we can construct a hereditary class that contains only the simple permutations $\{1, 12, 21, 2413, 25314\}$, where the growth of the principal Möbius function is exponential, answering questions in Burstein et al [3] and Jelínek et al [4].

We start by recalling some essential definitions and notation in Section 2, where we also provide some extensions of existing results. We formally define a 2413-balloon in Section 3, and we provide some results which will be used in the remainder of this paper. In Section 4, we derive an expression for the value of $\mu[\pi]$ when π is a double 2413-balloon, and following this we show that the growth of the Möbius function is exponential in Section 5. We return to the topic of 2413-balloons in Section 6, and derive an expression for the value of $\mu[\pi]$ when π is any 2413-balloon. Finally, we discuss the generalization of the balloon operator in Section 7. We also ask some questions regarding the growth of the Möbius function.

2 Essential definitions, notation, and results

In this section we recall some standard definitions and notation that we will use, and add some simple definitions and consequences of known results.

An *interval* in a permutation π is a contiguous set of indexes $i, i+1, \ldots, j$ such that the set of values $\{\pi_i, \pi_{i+1}, \ldots, \pi_j\}$ is also contiguous. Every permutation π has intervals of length 1 and of length $|\pi|$, which we call *trivial intervals*. A *simple* permutation is a permutation that only has trivial intervals. As examples, 1324 is not simple, as, for example, the second and third points (32) form a non-trivial interval, whereas 2413 is simple.

Given two permutations α and β, with lengths a and b respectively, the *direct sum* of α and β, written $\alpha \oplus \beta$ is the permutation $\alpha_1, \ldots, \alpha_a, \beta_1 + a, \ldots, \beta_b + a$. The *skew sum*, $\alpha \ominus \beta$, is the permutation $\alpha_1 + b, \ldots, \alpha_a + b, \beta_1, \ldots, \beta_b$.

Let α be a permutation, and r a positive integer. Then $\oplus^r \alpha$ is $\alpha \oplus \alpha \oplus \ldots \oplus \alpha \oplus \alpha$, with r occurrences of α.

If π is a permutation with length n, then the number of *corners* of π is the number of points of π that are extremal in both position and value, that is, $\pi_1 \in \{1, n\}$ or $\pi_n \in \{1, n\}$. It is easy to see that any permutation with length 2 or more can have at most two corners. We adopt the convention that the permutation 1 has one corner.

If a permutation π can be written as $1 \oplus 1 \oplus \tau$, $1 \ominus 1 \ominus \tau$, $\tau \oplus 1 \oplus 1$, or $\tau \ominus 1 \ominus 1$, then we say that π has a *long corner*.

We now have

Lemma 1. *If π has a long corner, then $\mu[\pi] = 0$.*

Lemma 2. *If π can be written as $\pi = 1 \oplus \tau$, or $\pi = \tau \oplus 1$ or $\pi = 1 \ominus \tau$ or $\pi = \tau \ominus 1$, and does not have a long corner, then $\mu[\pi] = -\mu[\tau]$.*

These are well-known consequences of Propositions 1 and 2 of Burstein, Jelínek, Jelínková and Steingrímsson [3], and we refrain from providing proofs here. The reader is directed to Lemma 4 in [2] for a proof of Lemma 1. Lemma 2 is a trivial extension of Corollary 3 in [3].

A *chain* in a poset interval $[1, \pi]$ is, for our purposes, a subset of the permutations in the interval $[1, \pi]$, where the subset includes the elements 1 and π, and any two elements of the subset are comparable. This last clause means that the subset has a total order. If a chain c has C elements, then we say that the length of c, written $|c|$, is $C - 1$.

Philip Hall's Theorem[6, Proposition 3.8.5] says that

$$\mu[\sigma, \pi] = \sum_{c \in \mathcal{C}(\sigma, \pi)} (-1)^{|c|} = \sum_{i=1}^{|\pi|-1} (-1)^i K_i$$

where $\mathcal{C}(\sigma, \pi)$ is the set of chains in the poset interval $[\sigma, \pi]$ which contain both σ and π, and K_i is the number of chains of length i.

If $\mathcal{C}$ is a subset of the chains in some poset interval $[\sigma, \pi]$, then the *Hall sum* of $\mathcal{C}$ is $\sum_{c \in \mathcal{C}} (-1)^{|c|}$.

A *parity-reversing involution*, $\Phi : \mathcal{C} \mapsto \mathcal{C}$, is an involution such that for any $c \in \mathcal{C}$, the parities of c and $\Phi(c)$ are different.

A simple corollary to Hall's Theorem is

Corollary 3. *If we can find a set of chains $\mathcal{C}$ with a parity-reversing involution, then the Hall sum of $\mathcal{C}$ is zero.*

Proof. Because there is a parity-reversing involution, the number of chains in $\mathcal{C}$ with odd length is equal to the number of chains with even length, so $\sum_{c\in\mathcal{C}}(-1)^{|c|} = 0$. □

We can also use Hall's Theorem if we have a subset of chains that meet a specific criteria:

Lemma 4. *Let π be any permutation with length three or more. Let ψ be a permutation with $1 < \psi < \pi$. Let $\mathcal{C}$ be the subset of chains in the poset interval $[1, \pi]$ where the second-highest element is ψ. Then*

$$\sum_{c\in\mathcal{C}}(-1)^{|c|} = -\mu[\psi].$$

Proof. If we remove π from the chains in $\mathcal{C}$, then we have all of the chains in the poset interval $[1, \psi]$, and the Hall sum of these chains is, by definition, $\mu[\psi]$. It follows that the Hall sum of the chains in $\mathcal{C}$ is $-\mu[\psi]$. □

Corollary 5. *Given a permutation π, and a set of permutations S where every $\sigma \in S$ satisfies $1 < \sigma < \pi$, then if $\mathcal{C}$ is the set of chains in the poset interval $[1, \pi]$ where the second-highest element is in S, then the Hall sum of $\mathcal{C}$ is $-\sum_{\sigma\in S}\mu[\sigma]$.*

Proof. First, partition $\mathcal{C}$ based on the second-highest element, and then apply Lemma 4 to each partition. □

When discussing chains, in general we will only be interested in a small subset of the chain containing two or three elements. We say that a *segment* of some chain c is a non-empty subset of the elements in c with the property that any element not in the segment is either less than every element in the segment, or is greater than every element in the segment.

In our proofs, given a set of chains $\mathcal{C}$, and a chain $c \in \mathcal{C}$, we will frequently want to construct a chain c' by using a parity-reversing involution Φ. Strictly speaking, Φ is a function that maps a set of permutations (which is a chain) to a set of permutations (which may not be a chain). As examples, if $\Phi(c)$ removes the largest or smallest element of c, or adds an element so that $\Phi(c)$ does not have a total order, then $\Phi(c)$ is not a chain. To show that Φ is a parity-reversing involution we will need to show that $\Phi(c)$ is a chain in $\mathcal{C}$, and that c and $\Phi(c)$ have opposite parities. In our discussions, we will typically set $c' = \Phi(c)$, and then show that the set of permutations c' is a chain. We will then, without further comment, treat c' as a chain.

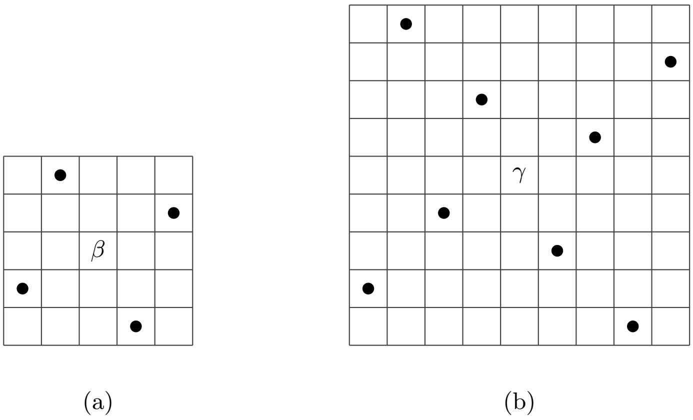

Figure 1: (a) The 2413-balloon $2413 \odot \beta$ and (b) the double 2413-balloon $2413 \odot 2413 \odot \gamma$.

3 2413-Balloons

In this section we define the vocabulary and notation specific to this paper. We also present some general results which will be used in later sections.

Given a non-empty permutation β, the *2413-balloon* of β is the permutation formed by inserting β into the centre of 2413, which we write as $2413 \odot \beta$. Formally, we have

$$(2413 \odot \beta)_i = \begin{cases} 2 & \text{if } i = 1 \\ |\beta| + 4 & \text{if } i = 2 \\ \beta_{i-2} + 2 & \text{if } i > 2 \text{ and } i \leq |\beta| + 2 \\ 1 & \text{if } i = |\beta| + 3 \\ |\beta| + 3 & \text{if } i = |\beta| + 4 \end{cases}$$

Figure 1(a) shows $2413 \odot \beta$.

The balloon operation as defined has to be right-associative and the definition given does not support overriding right-associativity. In other words, $2413 \odot 2413 \odot \beta$ must be $2413 \odot (2413 \odot \beta)$, and $(2413 \odot 2413) \odot \beta$ is not defined. In Section 7 we suggest how the balloon operation could be generalized.

Given some $\pi = 2413 \odot \beta$, if β is itself a 2413-balloon, so $\pi = 2413 \odot 2413 \odot \gamma$, then we say that π is a *double 2413-balloon*. Figure 1(b) shows a double 2413-balloon.

If we have $\pi = 2413 \odot \beta$, and we have some σ that is contained in π, we will frequently want to consider how σ can be represented in terms of sub-permutations of 2413 and β. We start by colouring the extremal points of π red, and all remaining points black. Note that the red points are a 2413 permutation, and the black points are β.

Now consider a specific embedding of σ into π. If the embedding is monochromatic then we require no special notation. Now assume that the embedding is not monochromatic. We set τ to be the permutation formed by the black points. If all four red points are used, then we now write $\sigma = 2413 \circledcirc \tau$. If some red points are unused, then we take 2413, and mark the red points that are unused with an overline, and then write σ using our balloon notation. As an example of this, if $\pi = 2413 \circledcirc 21 = 264315$, and $\sigma = 21$, then we could represent σ as $\overline{2}4\overline{13} \circledcirc 1$, $\overline{24}1\overline{3} \circledcirc 1$, or 21. Plainly, it is possible for σ to have multiple representations. We will see later how to handle this.

We will also want to consider permutations that are contained in $\pi = 2413 \circledcirc \beta$ where we know that some, but not all, red points have been deleted, but we do not want to explicitly specify which red points have been deleted. We will see that where we are considering these permutations, we will know that the embedding is not monochromatic. As before, we set τ to be the permutation formed by the black points. We then write $\sigma = \overline{2413}^{a} \circledcirc \tau$. The parameter a indicates that at least one and at most three red points have been deleted, but we do not know exactly which points. While this means that the permutation σ is not explicitly defined, we will see that this does not matter in practice, as the arguments we will use will apply to all permutations that match $\overline{2413}^{a} \circledcirc \tau$. As an example of how we will use this notation, and why we do not need to know exactly which red points are unused, we can write $\beta < \overline{2413}^{a} \circledcirc \beta < 2413 \circledcirc \beta$ and this is true for any β, and for any set of deleted points a.

If we have $\pi = 2413 \circledcirc \beta$, and $\sigma < \pi$ can be written in the form $\overline{2413}^{a} \circledcirc \beta$, then we say that σ is a *reduction* of π. The set of permutations that are reductions of π is written as R_{π}. Figure 2 shows all the reductions R_{π} of $\pi = 2413 \circledcirc \beta$. We can clearly see that if σ is a reduction of π, then there is a unique representation of σ in the form $\overline{2413}^{a} \circledcirc \beta$. Further, it is easy to see that a reduction of π cannot be written in the form $2413 \circledcirc \tau$.

The strategy that we will use in Sections 4 and 6 is to partition the chains in the poset interval $[1, \pi]$ into four sets, $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$. We then show that there are parity-reversing involutions on the sets $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$, and therefore, by Corollary 3, the Hall sum for each of these sets is zero, and so $\mu[\pi]$ is given by the Hall sum of the set $\mathcal{R}$. Finally, we show that the Hall sum of $\mathcal{R}$ can be written in terms of $\mu[\beta]$.

The chains in $\mathcal{R}$ are those chains where the second-highest element is in a set of permutations that we call the *critical set*, written C_{π}. The members of the critical set, and hence the chains in $\mathcal{R}$, depend on the form of π, and so will be defined explicitly when we fix the form of π. In every case we have $\mathsf{C}_{\pi} \subseteq \mathsf{R}_{\pi} \cup \{\beta\}$, and so we can see that for any permutation $\sigma \in \mathsf{C}_{\pi}$ we must have $|\sigma| \geq |\beta|$.

We have some results that are independent of C_{π}, and, once we have given some some further definitions, we present these in the current section to avoid repetition.

Let π be a 2413-balloon, and let c be any chain in the poset interval $[1, \pi]$.

Since the top of the chain is, by definition, a 2413-balloon, it follows that c has a unique maximal segment that includes the element π, where every element in the segment is a 2413-balloon. We call the smallest element in this segment the

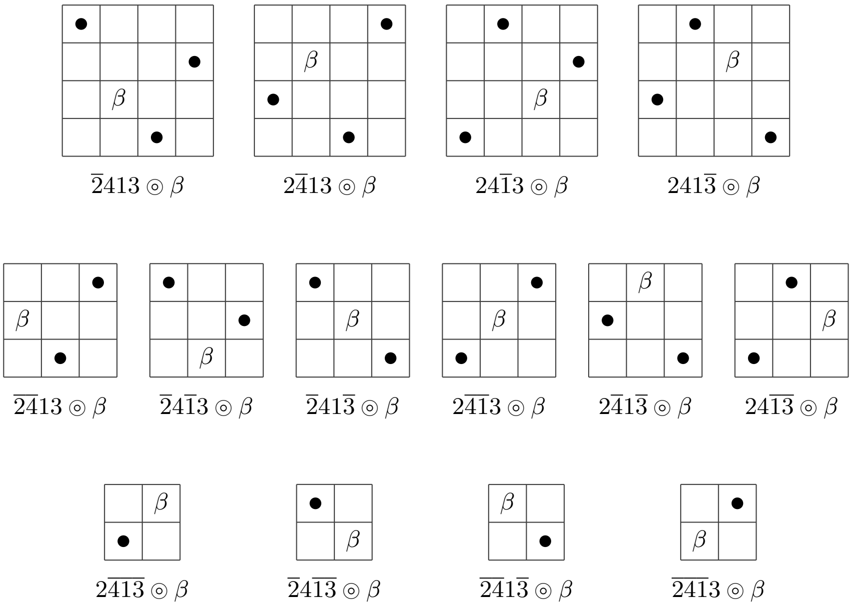

Figure 2: Reductions R_π of $\pi = 2413 \circledcirc \beta$.

least 2413-balloon[1].

Further, since the permutation 1 is not a 2413-balloon, it follows that c has an element that is immediately below the least 2413-balloon in the chain, and we call this element the *pivot*.

We define ϕ_c to be the least 2413-balloon in c, ψ_c to be the pivot in c, τ_c to be the permutation that satisfies $2413 \circledcirc \tau_c = \phi_c$, and κ_c to be the second-highest element of c. Note that ϕ_c and ψ_c must be distinct, but we can have $\tau_c = \psi_c$. Further, κ_c is independent, and may be the same as ϕ_c, ψ_c or τ_c. Figure 3 shows some example chains, highlighting these elements.

If every representation of ψ_c uses only extremal points from ϕ_c, then we have $\psi_c \leq 2413$. Similarly, if every representation of ψ_c uses only non-extremal points from ϕ_c, then we have $\psi_c \leq \tau_c$.

If neither of these cases is true, that is, $\psi_c \not\leq 2413$ and $\psi_c \not\leq \tau_c$, then any representation of ψ_c uses at least one extremal point and one non-extremal point from ϕ_c, and it follows that there is at least one permutation η such that we can write $\psi_c = \overline{2413}^a \circledcirc \eta$, with $\eta \leq \tau_c$. If there is more than one possibility for η, then we choose η so that $|\eta|$ is maximal. If there is still more than one possibility, then we choose η so that it is lexicographically smallest. We note here that this last choice is, essentially, arbitrary, as the requirement is simply that we have a deterministic way to choose a unique maximal η. We also note that we will modify this algorithm slightly in Section 6.

[1] The name should really be "least 2413-balloon in the chain that has only 2413-balloons above it".

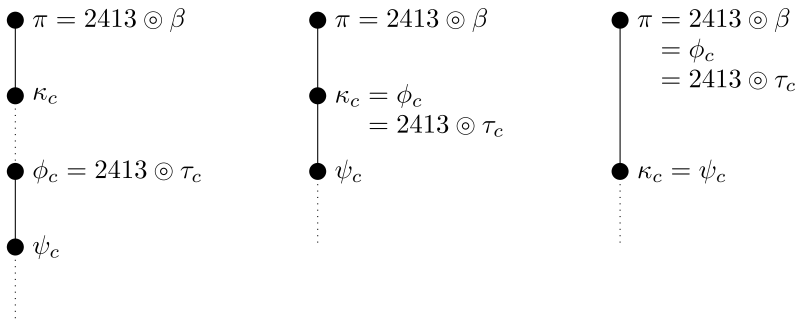

Figure 3: Examples of chains, showing some possible relationships between π, κ_c, ϕ_c, and ψ_c.

We now say that if $\kappa_c \notin \mathsf{C}_\pi$, and $\psi_c \not\leq 2413$, and $\psi_c \not\leq \tau_c$, then η_c is the maximal (in the sense defined above) permutation that satisfies $\psi_c = \overline{2413}^a \odot \eta_c$, with $\eta_c \leq \tau_c$.

We are now in a position to give a definition of the sets $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$. This definition depends on the critical set C_π, which, as stated earlier, differs slightly between proofs, although we always have $\mathsf{C}_\pi \subseteq \mathsf{R}_\pi \cup \beta$.

Let $\mathcal{C}$ be the set of chains in the poset interval $[1, \pi]$. We define subsets of $\mathcal{C}$ as follows:

$$\begin{aligned}
\mathcal{R} &= \{c : c \in \mathcal{C} \text{ and } \kappa_c \in \mathsf{C}_\pi\}, \\
\mathcal{G} &= \{c : c \in \mathcal{C} \setminus \mathcal{R} \text{ and } \psi_c \leq 2413\}, \\
\mathcal{B} &= \{c : c \in \mathcal{C} \setminus (\mathcal{R} \cup \mathcal{G}) \text{ and } \psi_c \leq \tau_c\}, \\
\mathcal{Y} &= \{c : c \in \mathcal{C} \setminus (\mathcal{R} \cup \mathcal{G} \cup \mathcal{B})\}.
\end{aligned}$$

Clearly, every chain in $[1, \pi]$ is included in exactly one of these subsets, and so these sets are a partition of the chains.

We now define three functions, one for each of $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$, which will give us parity-reversing involutions.

$$\Phi_{\mathcal{G}}(c) = \begin{cases} c \setminus \{2413\} & \text{If } \psi_c = 2413 \\ c \cup \{2413\} & \text{If } \psi_c < 2413 \end{cases}$$

$$\Phi_{\mathcal{B}}(c) = \begin{cases} c \setminus \{2413 \odot \psi_c\} & \text{If } \psi_c = \tau_c \\ c \cup \{2413 \odot \psi_c\} & \text{If } \psi_c < \tau_c \end{cases}$$

$$\Phi_{\mathcal{Y}}(c) = \begin{cases} c \setminus \{2413 \odot \eta_c\} & \text{If } \eta_c = \tau_c \\ c \cup \{2413 \odot \eta_c\} & \text{If } \eta_c < \tau_c \end{cases}$$

For $\Phi_{\mathcal{G}}(c)$ to be a parity-reversing involution on $\mathcal{G}$, we need to show that if $c \in \mathcal{G}$, then $\Phi_{\mathcal{G}}(c)$ is a chain, that $\Phi(c) \in \mathcal{G}$, and that c and $\Phi(c)$ have different parities. It is easy to see that this last condition is true. Similar comments apply to $\Phi_{\mathcal{B}}(c)$ and $\mathcal{B}$, and $\Phi_{\mathcal{Y}}(c)$ and $\mathcal{Y}$.

For $\Phi_{\mathcal{G}}(c)$ we can show that all the conditions hold for any critical set. For $\Phi_{\mathcal{B}}(c)$ and $\Phi_{\mathcal{Y}}(c)$ we show that some weaker conditions hold for an arbitrary critical set, and then, when we have an explicit critical set, we show that all conditions hold. The following Lemma gives us a result that applies to $\Phi_{\mathcal{G}}(c)$, $\Phi_{\mathcal{B}}(c)$, and $\Phi_{\mathcal{Y}}(c)$ for any critical set $\mathsf{C}_\pi \subseteq \mathsf{R}_\pi \cup \{\beta\}$, and we will use this result in both Section 4 and Section 6.

Lemma 6. *Let $\pi = 2413 \circledcirc \beta$, with $|\beta| > 4$, and let $\mathcal{R}$, and $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$ be as defined above.*

(a) If $c \in \mathcal{G}$, then $\Phi_{\mathcal{G}}(c) \in \mathcal{G}$.

(b) If $c \in \mathcal{B}$, with $\psi_c = \tau_c$, and $\Phi_{\mathcal{B}}(c)$ is a chain, then $\Phi_{\mathcal{B}}(c) \in \mathcal{B} \cup \mathcal{R}$

(c) If $c \in \mathcal{B}$, with $\psi_c < \tau_c$, then $\Phi_{\mathcal{B}}(c) \in \mathcal{B} \cup \mathcal{R}$

(d) If $c \in \mathcal{Y}$, with $\eta_c = \tau_c$, and $\Phi_{\mathcal{Y}}(c)$ is a chain, then $\Phi_{\mathcal{Y}}(c) \in \mathcal{Y} \cup \mathcal{R}$

(e) If $c \in \mathcal{Y}$, with $\eta_c < \tau_c$, then $\Phi_{\mathcal{Y}}(c) \in \mathcal{Y} \cup \mathcal{R}$

Proof. **Case (a).** First, assume that $c \in \mathcal{G}$ with $\psi_c = 2413$. Then c contains a segment $2413 < 2413 \circledcirc \tau$, and $c' = \Phi_{\mathcal{G}}(c) = c \setminus \{2413\}$. We can see that c' is a chain, as 2413 is neither the smallest nor the largest entry in c'. Further, $\psi_{c'} < 2413$. Since $|\beta| > 4$, and $|\psi_{c'}| < 4$ we must have $c' \notin \mathcal{R}$, and therefore $c' \in \mathcal{G}$.

Now assume that $c \in \mathcal{G}$ with $\psi_c < 2413$. Then c contains a segment $\psi_c < 2413 \circledcirc \gamma$, and $c' = \Phi_{\mathcal{G}}(c) = c \cup \{2413\}$. We can see that c' is a chain, since $\psi_c < 2413 < 2413 \circledcirc \gamma$, and further, $\psi_{c'} = 2413$. Since $|\beta| > 4$, and $|\psi_{c'}| = 4$ we must have $c' \notin \mathcal{R}$, and therefore $c' \in \mathcal{G}$.

Case (b). Let c be a chain in $\mathcal{B}$. Assume first that $\psi_c = \tau_c$. Then c contains a segment $\psi_c < 2413 \circledcirc \tau_c$, and $c' = \Phi_{\mathcal{B}}(c) = c \setminus \{2413 \circledcirc \tau_c\}$. If $\tau_c = \beta$, then c' is not a chain, so we must have $\tau_c < \beta$, and therefore c' is a chain that contains a segment $\psi_c < 2413 \circledcirc \gamma$, with $\tau_c < \gamma$, so $\psi_c < \gamma$. Now, ψ_c is the pivot of c', so we cannot have $c' \in \mathcal{G}$ as this would imply that $c \in \mathcal{G}$, which is a contradiction. Thus either $c' \in \mathcal{R}$ or $c' \in \mathcal{B}$.

Case (c). Now assume that $\psi_c < \tau_c$. Then c contains a segment $\psi_c < 2413 \circledcirc \tau_c$, and $c' = \Phi_{\mathcal{B}}(c) = c \cup \{2413 \circledcirc \psi_c\}$. We can see that c' is a chain since $\psi_c < 2413 \circledcirc \psi_c < 2413 \circledcirc \tau_c$. Now, ψ_c is the pivot of c', so we cannot have $c' \in \mathcal{G}$ as this would imply that $c \in \mathcal{G}$, which is a contradiction. Since $2413 \circledcirc \psi_c$ is the least 2413-balloon in c', we can see that either $c' \in \mathcal{R}$ or $c' \in \mathcal{B}$.

Case (d). Let c be a chain in $\mathcal{Y}$. First, assume that $\eta_c = \tau_c$. Then c contains a segment $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \tau_c$, and $c' = \Phi_{\mathcal{Y}}(c) = c \setminus \{2413 \circledcirc \tau_c\}$. If $\tau_c = \beta$ then c' is not a chain, so we must have $\tau_c < \beta$, and so in c we must have a segment $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \tau_c < 2413 \circledcirc \gamma$, with $\tau_c < \gamma$, and thus in c' we have a segment $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \gamma$. Now, $\overline{2413}^a \circledcirc \eta_c$ is the pivot of c', so we cannot have $c' \in \mathcal{G}$ as this would imply that $c \in \mathcal{G}$, which is a contradiction. To see that $c' \notin \mathcal{B}$, see that this would imply that $\overline{2413}^a \circledcirc \eta_c < \gamma$, and this in turn implies $\overline{2413}^a \circledcirc \eta_c < \tau_c$, which is a contradiction. Thus either $c' \in \mathcal{R}$ or $c' \in \mathcal{Y}$.

Case (e). Finally, assume that $\eta_c < \tau_c$. Then c contains a segment $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \tau_c$, and $c' = \Phi_{\mathcal{Y}}(c) = c \cup \{2413 \circledcirc \eta_c\}$, so c' is a chain since $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \eta_c < 2413 \circledcirc \tau_c$. Now, $\overline{2413}^a \circledcirc \eta_c$ is the pivot of c', so we cannot have $c' \in \mathcal{G}$ as this would imply that $c \in \mathcal{G}$, which is a contradiction. To see that $c' \notin \mathcal{B}$, note that this would require $\overline{2413}^a \circledcirc \eta_c < \eta_c$, and this is clearly impossible, so either $c' \in \mathcal{R}$ or $c' \in \mathcal{Y}$. □

We now have

Observation 7. *If $\pi = 2413 \circledcirc \beta$, then to show that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$ and that $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$ it is sufficient to show that:*

(a) If $c \in \mathcal{B}$ and $\psi_c = \tau_c$, then $\Phi_{\mathcal{B}}(c)$ is a chain

(b) If $c \in \mathcal{B}$, $\Phi_{\mathcal{B}}(c) \notin \mathcal{R}$.

(c) If $c \in \mathcal{Y}$ and $\eta_c = \tau_c$, then $\Phi_{\mathcal{Y}}(c)$ is a chain.

(d) If $c \in \mathcal{Y}$, $\Phi_{\mathcal{Y}}(c) \notin \mathcal{R}$.

Further, if $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$, and $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$, then $\mu[\pi] = -\sum_{\sigma \in \mathsf{C}_\pi} \mu[\sigma]$.

Proof. Combining (a) and (b) above with cases (b) and (c) of Lemma 6 gives us that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$. Similarly, combining (c) and (d) with cases (d) and (e) of Lemma 6 gives us that $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$.

This now gives us that $\sum_{c \in \mathcal{B}} (-1)^{|c|} = 0$, and $\sum_{c \in \mathcal{Y}} (-1)^{|c|} = 0$, and from Lemma 6 we have $\sum_{c \in \mathcal{G}} (-1)^{|c|} = 0$, and so we must have $\mu[\pi] = \sum_{c \in \mathcal{R}} (-1)^{|c|}$. Since the chains in $\mathcal{R}$ are defined by the second-highest element (κ_c) being in the critical set, the final part of the observation follows by applying Corollary 5. □

4 The Möbius function of double 2413-balloons

We are now able to state and prove our first major result.

Theorem 8. *Let $\pi = 2413 \circledcirc \beta$, where β is a 2413-balloon, Then $\mu[\pi] = 2\mu[\beta]$.*

Proof. Let $\mathsf{C}_\pi = \mathsf{R}_\pi$, so the critical set is simply the reductions of π.

Using Observation 7, we will show that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$ and $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$. Once we have shown that we have parity-reversing involutions, we will then show how to express the Hall sum of $\mathcal{R}$ in terms of $\mu[\beta]$.

Proof that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$. Let c be a chain in $\mathcal{B}$.

First, assume that $\psi_c = \tau_c$. Then the segment $\psi_c < \phi_c$ can be written $\tau_c < 2413 \circledcirc \tau_c$. If $\tau_c = \beta$, then the pivot is β, but this is a contradiction, since β is a 2413-balloon, so we must have $\tau_c < \beta$. Let $c' = \Phi_{\mathcal{B}}(c) = c \setminus \{2413 \circledcirc \tau_c\}$. Since $\tau_c < \beta$, c' is a chain. We now show that $c' \notin \mathcal{R}$. Assume, to the contrary, that $c' \in \mathcal{R}$ which implies that τ_c is a reduction of π. Now, $\tau_c < \beta$, so $|\tau_c| < |\beta|$. If τ_c is a reduction of π, then $|\tau_c| > |\beta|$, which is a contradiction, so τ_c is not a reduction of π, therefore $c' \notin \mathcal{R}$.

Now assume that $\psi_c < \tau_c$. Let $c' = \Phi_{\mathcal{B}}(c) = c \cup 2413 \circledcirc \psi_c$. We know by Lemma 6 that this is a chain. Either $\kappa_c = \kappa_{c'}$, or $\kappa_{c'}$ is a 2413-balloon. If $\kappa_c = \kappa_{c'}$, then $c' \notin \mathcal{R}$. If $\kappa_{c'}$ is a 2413-balloon, then $\kappa_{c'} \notin \mathsf{C}_\pi$, so $c' \notin \mathcal{R}$. Thus we must have $c' \notin \mathcal{R}$.

So now we have that if $c \in \mathcal{B}$ and $\psi_c = \tau_c$, then $\Phi_{\mathcal{B}}(c)$ is a chain; and that for any $c \in \mathcal{C}$, $\Phi_{\mathcal{B}}(c) \in \mathcal{B}$. It follows that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$. ◻

Proof that $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$. Let c be a chain in $\mathcal{Y}$. Recall that the segment $\psi_c < \phi_c$ can be written as $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \tau_c$.

First, assume that $\eta_c = \tau_c$, so the segment $\psi_c < \phi_c$ is $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \eta_c$. If $\eta_c = \beta$, then the second-highest element of the chain would be a reduction of π, which is a contradiction, so we must have $\eta_c < \beta$. Let $c' = \Phi_{\mathcal{Y}}(c) = c \setminus \{2413 \circledcirc \eta_c\}$. This is a valid chain since $2413 \circledcirc \eta_c < \pi$. To see that $c' \notin \mathcal{R}$, assume to the contrary that $c' \in \mathcal{R}$, and see that this means that $\psi_c = \overline{2413}^a \circledcirc \eta_c$ must be a reduction of π. If ψ_c was a reduction of π, then we could write $\overline{2413}^a \circledcirc \eta_c = \overline{2413}^b \circledcirc \beta$. Now, $\eta_c < \beta$, so this means that η_c would not be maximal, which is a contradiction. It follows that $c' \notin \mathcal{R}$.

Now assume that $\eta_c \neq \tau_c$. Let $c' = \Phi_{\mathcal{Y}}(c) = c \cup \{2413 \circledcirc \eta_c\}$. We know by Lemma 6 that this is a chain. Either $\kappa_c = \kappa_{c'}$, or $\kappa_{c'}$ is a 2413-balloon. If $\kappa_c = \kappa_{c'}$, then $c' \notin \mathcal{R}$. If $\kappa_{c'}$ is a 2413-balloon, then $\kappa_{c'} \notin \mathsf{C}_\pi$, so $c' \notin \mathcal{R}$. Thus we must have $c' \in \mathcal{Y}$.

So now we have that if $c \in \mathcal{Y}$ and $\eta_c = \tau_c$, then $\Phi_{\mathcal{B}}(c)$ is a chain; and that for any $c \in \mathcal{Y}$, $\Phi_{\mathcal{Y}}(c) \in \mathcal{Y}$. It follows that $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$. ◻

We have now shown that $\Phi_{\mathcal{G}}$, $\Phi_{\mathcal{B}}$ and $\Phi_{\mathcal{Y}}$ are parity-reversing involutions on $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{Y}$ respectively. If follows from Observation 7 that $\mu[\pi] = -\sum_{\sigma \in \mathsf{C}_\pi} \mu[\sigma]$. We now show how to express $\mu[\sigma]$, where $\sigma \in \mathsf{C}_\pi$ in terms of $\mu[\beta]$.

First, take the case where $\sigma = \overline{2}413 \circledcirc \beta$, which is the first permutation in Figure 2. Note that we can write $\sigma = 1 \ominus ((\beta \ominus 1) \oplus 1)$. Applying Lemma 2 to the outermost three points in σ (those from the $\overline{2}413$), we find that $\mu[\sigma] = -\mu[\beta]$. The other cases are similar, and this gives us:[2]

[2]This table is slightly redundant, as the entries are determined by the parity of the "red" points. We include it as later results have similar tables where some values of $\mu[\sigma]$ are zero, and this gives a consistent presentation.

σ	$\mu[\sigma]$
$\bar{2}413 \circledcirc \beta$	$-\mu[\beta]$
$2\bar{4}13 \circledcirc \beta$	$-\mu[\beta]$
$24\bar{1}3 \circledcirc \beta$	$-\mu[\beta]$
$241\bar{3} \circledcirc \beta$	$-\mu[\beta]$

σ	$\mu[\sigma]$
$\overline{24}13 \circledcirc \beta$	$\mu[\beta]$
$\bar{2}4\bar{1}3 \circledcirc \beta$	$\mu[\beta]$
$\bar{2}41\bar{3} \circledcirc \beta$	$\mu[\beta]$
$2\overline{41}3 \circledcirc \beta$	$\mu[\beta]$
$2\bar{4}1\bar{3} \circledcirc \beta$	$\mu[\beta]$
$24\overline{13} \circledcirc \beta$	$\mu[\beta]$

σ	$\mu[\sigma]$
$2\overline{413} \circledcirc \beta$	$-\mu[\beta]$
$\bar{2}\overline{413} \circledcirc \beta$	$-\mu[\beta]$
$\overline{24}1\bar{3} \circledcirc \beta$	$-\mu[\beta]$
$\overline{241}3 \circledcirc \beta$	$-\mu[\beta]$

It is now easy to see that

$$\sum_{\sigma \in \mathsf{C}_\pi} \mu[\sigma] = -2\mu[\beta]$$

and the result follows directly. □

5 The growth of the Möbius function

We define $\max_\mu(n) = \max\{|\mu[\pi]| : |\pi| = n\}$. Previous work in [4] and [5] has shown that the growth of $\max_\mu(n)$ is at least polynomial. We will show that the growth is at least exponential. We have

Theorem 9. *For all n,* $\max_\mu(n) \geq 2^{\lfloor n/4 \rfloor - 1}$.

Proof. We start by a defining a function to construct a permutation of length n.

$$\pi^{(n)} = \begin{cases} 1 & \text{If } n = 1 \\ 12 & \text{If } n = 2 \\ 132 & \text{If } n = 3 \\ 2413 & \text{If } n = 4 \\ 2413 \circledcirc \pi^{(n-4)} & \text{Otherwise} \end{cases}$$

Note that for $n > 8$, $\pi^{(n)}$ is a double 2413-balloon. It is simple to calculate $\mu[\pi^{(n)}]$ for $n = 1, \ldots, 8$, and these values are given below.

$$\begin{aligned} \mu[\pi^{(1)}] &= 1, & \mu[\pi^{(5)}] &= 4, \\ \mu[\pi^{(2)}] &= -1, & \mu[\pi^{(6)}] &= -1, \\ \mu[\pi^{(3)}] &= 1, & \mu[\pi^{(7)}] &= 1, \\ \mu[\pi^{(4)}] &= -3, & \mu[\pi^{(8)}] &= -6. \end{aligned}$$

These values match Theorem 9, and so this is true for $n \leq 8$. For $n > 8$, $\mu[\pi^{(n)}] = 2\mu[\pi^{(n-4)}]$ by Theorem 8, and the result follows immediately. □

Remark 10. It is easy to see that, with the definitions above, the only simple permutations that can be contained in $\pi^{(n)}$ are 1, 12, 21, 2413, and 25314. This answers Problem 4.4 in [4], which asks whether $\mu[\pi]$ is bounded on a hereditary class which contains only finitely many simple permutations, as, by Theorem 9, we have unbounded growth, but only finitely many simple permutations.

6 The Möbius function of 2413-balloons

Theorem 8 gives us an expression for the value of the Möbius function $\mu[\pi]$ when π is a double-2413-balloon. We expand on this to find an expression for the Möbius function $\mu[\pi]$ when π is any 2413-balloon.

We start with a Lemma that handles the case where β is not a 2413-balloon, and has more than four points. The structure of our proof is similar to that of Lemma 8, but we present a complete argument to aid readability.

We will show

Lemma 11. *Let $\pi = 2413 \odot \beta$, where β is not a 2413-balloon, and $|\beta| > 4$. Then $\mu[\pi] = \mu[\beta]$.*

Proof. If β has one corner, then without loss of generality, we can assume by symmetry that $\beta = 1 \oplus \gamma$. Similarly, if β has two corners, then we can assume that $\beta = 1 \oplus \gamma \oplus 1$.

As before, we will use Observation 7. We will show that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$ and $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$. Once we have shown that we have parity-reversing involutions, we will then show how to express the Hall sum of $\mathcal{R}$ in terms of $\mu[\beta]$.

One problem we encounter when β has at least one corner is that the unique representation defined in Section 3 may not work. As an example of how this representation fails, assume that β is not in the critical set, that β has one corner, and consider the chains that end $\beta < 2413 \odot \beta$. These chains would normally be in $\mathcal{B}$, and $\Phi_{\mathcal{B}}$ would remove $2413 \odot \beta$, which does not give a chain. To overcome this specific problem, we can represent β as $2\overline{413} \odot \gamma$. We move these chains from $\mathcal{B}$ to $\mathcal{Y}$, and then $\Phi_{\mathcal{Y}}$ adds $2413 \odot \gamma$. We may also need to adjust the critical set.

The critical set depends on the number of corners of β, as defined below.

Corners in β	Critical set
No corners	$\mathsf{C}_\pi = \mathsf{R}_\pi \cup \{\beta\}$
One corner	$\mathsf{C}_\pi = \mathsf{R}_\pi \setminus \{\overline{24}13 \odot \beta,\ \overline{24}1\overline{3} \odot \beta,\ \overline{2413} \odot \beta\}$
Two corners	$\mathsf{C}_\pi = \mathsf{R}_\pi$

If β has one corner, then the permutations $\overline{24}13 \odot \beta$, $\overline{24}1\overline{3} \odot \beta$, and $\overline{2413} \odot \beta$ are represented as $2\overline{4}13 \odot \gamma$, $2\overline{4}1\overline{3} \odot \gamma$, and $2\overline{413} \odot \gamma$ respectively. If β has two corners, then β is represented as $2\overline{413} \odot \gamma$. In all other cases, we use the representation as described in Section 3. It is routine to check that Lemma 6 is still valid with these changes.

Proof that $\Phi_{\mathcal{B}}$ is a parity-reversing involution on $\mathcal{B}$. Let c be a chain in $\mathcal{B}$.

First, assume that $\psi_c = \tau_c$. Then the segment $\psi_c < \phi_c$ can be written $\tau_c < 2413 \odot \tau_c$. If β has zero or one corners, and $\tau_c = \beta$, then the pivot is β, but then $c \in \mathcal{R}$, which is a contradiction. If β has two corners, and $\psi_c = \beta$, then $c \in \mathcal{Y}$ as this is the special case. We conclude that we must have $\tau_c < \beta$. Let

$c' = \Phi_{\mathcal{B}}(c) = c \setminus \{2413 \circledcirc \tau_c\}$. Since $\tau_c < \beta$, c' is a valid chain. We now show that $c' \notin \mathcal{R}$. Assume, to the contrary, that $c' \in \mathcal{R}$ which implies that τ_c is a reduction of π or $\tau_c = \beta$. Now, $\tau_c < \beta$, so $|\tau_c| < |\beta|$. If τ_c is a reduction of π or is β, then $|\tau_c| \geq |\beta|$, which is a contradiction, so τ_c is not a reduction of π or β, therefore $c' \notin \mathcal{R}$.

Now assume that $\psi_c < \tau_c$. Let $c' = \Phi_{\mathcal{B}}(c) = c \cup \{2413 \circledcirc \psi_c\}$, and we know from Lemma 6 that c' is a chain. Either $\kappa_c = \kappa_{c'}$, or $\kappa_{c'}$ is a 2413-balloon, but in either case we have $c' \notin \mathcal{R}$.

So if $c \in \mathcal{B}$, then $\Phi_{\mathcal{B}}(c)$ is a chain in $\mathcal{B}$, and thus $\Phi_{\mathcal{B}}$ is a parity-reversing involution. ⊡

Proof that $\Phi_{\mathcal{Y}}$ is a parity-reversing involution on $\mathcal{Y}$. Let c be a chain in $\mathcal{Y}$. Recall that the segment $\psi < \phi$ can be written as $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \tau_c$.

First, assume that $\eta_c = \tau_c$, so the segment $\psi_c < \phi_c$ is $\overline{2413}^a \circledcirc \eta_c < 2413 \circledcirc \eta_c$. If $\eta_c = \beta$, then, taking into account the exceptional cases, $\kappa_c \in \mathsf{C}_\pi$, which is a contradiction, so we must have $\eta_c < \beta$. Let $c' = \Phi_{\mathcal{Y}}(c) = c \setminus \{2413 \circledcirc \eta_c\}$. This is a chain since $2413 \circledcirc \eta_c < \pi$. To see that $c' \notin \mathcal{R}$, assume otherwise. First, assume that $\overline{2413}^a \circledcirc \eta_c$ is a reduction of π in C_π Then we can write $\overline{2413}^a \circledcirc \eta_c = \overline{2413}^b \circledcirc \beta$. Now, since $\eta_c < \beta$, η_c would not be maximal, which is a contradiction, therefore $\overline{2413}^a \circledcirc \eta_c$ is not a reduction of π in C_π. Now assume that $\overline{2413}^a \circledcirc \eta_c = \beta$, and β has no corners. This is a contradiction, since $\overline{2413}^a \circledcirc \eta_c$ has a corner, so $\overline{2413}^a \circledcirc \eta_c \neq \beta$. If β has at least one corner, then $\beta \notin \mathsf{C}_\pi$. Thus we have that $c' \notin \mathcal{R}$.

Now assume that $\eta_c \neq \tau_c$. Let $c' = \Phi(c) = c \cup \{2413 \circledcirc \eta_c\}$. We know, from Lemma 6, that c' is a chain. Either $\kappa_c = \kappa_{c'}$, or $\kappa_{c'}$ is a 2413-balloon, but in either case we have $c' \notin \mathcal{R}$.

So if $c \in \mathcal{Y}$, then $\Phi_{\mathcal{Y}}(c)$ is a valid chain in $\mathcal{Y}$, and thus $\Phi_{\mathcal{Y}}$ is a parity-reversing involution. ⊡

We have shown that $\Phi_{\mathcal{G}}$, $\Phi_{\mathcal{B}}$ and $\Phi_{\mathcal{Y}}$ are parity-reversing involutions on $\mathcal{G}$, $\mathcal{B}$ and $\mathcal{Y}$ respectively. It follows from Observation 7 that $\mu[\pi] = -\sum_{\sigma \in \mathsf{C}_\pi} \mu[\sigma]$. We now show how to express $\mu[\sigma]$, where $\sigma \in \mathsf{C}_\pi$ in terms of $\mu[\beta]$. We use a similar mechanism to that used in Theorem 8. There are some additional considerations where β has one or two corners. As an example, take the case where $\sigma = \overline{2413} \circledcirc \beta$, and β has one corner, and so, by our assumption, can be written as $1 \oplus \gamma$. We can write $\sigma = ((1 \oplus \beta) \ominus 1) \oplus 1$, and expanding β we have $\sigma = ((1 \oplus 1 \oplus \gamma) \ominus 1) \oplus 1$, Applying Lemma 2 to the outermost two points in σ, we find that $\mu[\sigma] = \mu[1 \oplus 1 \oplus \gamma]$, and by Lemma 1 we now have $\mu[\sigma] = 0$. Because of this, our analysis depends on the number of corners of β, and we consider each case separately below.

If β has no corners, then we have

σ	$\mu[\sigma]$
$\overline{2}413 \circledcirc \beta$	$-\mu[\beta]$
$2\overline{4}13 \circledcirc \beta$	$-\mu[\beta]$
$24\overline{1}3 \circledcirc \beta$	$-\mu[\beta]$
$241\overline{3} \circledcirc \beta$	$-\mu[\beta]$

σ	$\mu[\sigma]$
$\overline{24}13 \circledcirc \beta$	$\mu[\beta]$
$\overline{2}4\overline{1}3 \circledcirc \beta$	$\mu[\beta]$
$\overline{2}41\overline{3} \circledcirc \beta$	$\mu[\beta]$
$2\overline{41}3 \circledcirc \beta$	$\mu[\beta]$
$2\overline{4}1\overline{3} \circledcirc \beta$	$\mu[\beta]$
$24\overline{13} \circledcirc \beta$	$\mu[\beta]$

σ	$\mu[\sigma]$
$2\overline{413} \circledcirc \beta$	$-\mu[\beta]$
$\overline{2}\,\overline{413} \circledcirc \beta$	$-\mu[\beta]$
$\overline{24}1\overline{3} \circledcirc \beta$	$-\mu[\beta]$
$\overline{241}3 \circledcirc \beta$	$-\mu[\beta]$
β	$\mu[\beta]$

If β has one corner, under our assumption that $\beta = 1 \oplus \gamma$, we have

σ	$\mu[\sigma]$
$\overline{2}413 \circledcirc \beta$	$-\mu[\beta]$
$2\overline{4}13 \circledcirc \beta$	0
$24\overline{1}3 \circledcirc \beta$	$-\mu[\beta]$
$241\overline{3} \circledcirc \beta$	$-\mu[\beta]$

σ	$\mu[\sigma]$
$\overline{2}4\overline{1}3 \circledcirc \beta$	$\mu[\beta]$
$\overline{2}41\overline{3} \circledcirc \beta$	$\mu[\beta]$
$2\overline{41}3 \circledcirc \beta$	0
$2\overline{4}1\overline{3} \circledcirc \beta$	0
$24\overline{13} \circledcirc \beta$	$\mu[\beta]$

σ	$\mu[\sigma]$
$2\overline{413} \circledcirc \beta$	0
$\overline{2}\,\overline{413} \circledcirc \beta$	$-\mu[\beta]$

Finally, if β has two corners, under our assumption that $\beta = 1 \oplus \gamma \oplus 1$, we have

σ	$\mu[\sigma]$
$\overline{2}413 \circledcirc \beta$	$-\mu[\beta]$
$2\overline{4}13 \circledcirc \beta$	0
$24\overline{1}3 \circledcirc \beta$	0
$241\overline{3} \circledcirc \beta$	$-\mu[\beta]$

σ	$\mu[\sigma]$
$\overline{24}13 \circledcirc \beta$	$\mu[\beta]$
$\overline{2}4\overline{1}3 \circledcirc \beta$	0
$\overline{2}41\overline{3} \circledcirc \beta$	$\mu[\beta]$
$2\overline{41}3 \circledcirc \beta$	0
$2\overline{4}1\overline{3} \circledcirc \beta$	0
$24\overline{13} \circledcirc \beta$	$\mu[\beta]$

σ	$\mu[\sigma]$
$2\overline{413} \circledcirc \beta$	0
$\overline{2}\,\overline{413} \circledcirc \beta$	$-\mu[\beta]$
$\overline{24}1\overline{3} \circledcirc \beta$	$-\mu[\beta]$
$\overline{241}3 \circledcirc \beta$	0

In all three cases we have

$$\sum_{\sigma \in \mathsf{C}_\pi} \mu[\sigma] = -\mu[\beta]$$

and the result follows directly. □

We are now in a position to state and prove the main Theorem for this section.

Theorem 12. *Let* $\pi = 2413 \circledcirc \beta$. *Then*

$$\mu[\pi] = \begin{cases} 4 & \textit{If } \beta = 1 \\ -6 & \textit{If } \beta = 2413 \\ 2\mu[\beta] & \textit{If } \beta \textit{ is a 2413-balloon} \\ \mu[\beta] & \textit{Otherwise.} \end{cases}$$

Proof. The value of $\mu[2413 \circledcirc \beta]$ for the symmetry classes of β with $|\beta| \leq 4$ are shown below.

β	$\mu[\beta]$	$\mu[2413 \circledcirc \beta]$
1	1	4
12	-1	-1
123	0	0
132	1	1
1234	0	0
1243	0	0

β	$\mu[\beta]$	$\mu[2413 \circledcirc \beta]$
1324	-1	-1
1342	-1	-1
1432	0	0
2143	-1	-1
2413	-3	-6

It is easy to see that these values meet Theorem 12. We now combine Theorem 8 and Lemma 11 to complete the proof. □

7 Concluding remarks

7.1 Generalising the balloon operator

Given two permutations α and β, with lengths a and b respectively, and two integers i, j which satisfy $0 \leq i, j \leq a$, the *i, j-balloon* of β by α, written as $\alpha \circledcirc_{i,j} \beta$, is the permutation formed by inserting the permutation β into α between the i-th and $i+1$-th columns of α, and between the j-th and $j+1$-th rows of α. The integers i and j are, collectively, the *indexes* of the balloon.

Formally, we have

$$(\alpha \circledcirc_{i,j} \beta)_x = \begin{cases} \alpha_x & \text{if } x \leq i \text{ and } \alpha_x \leq j \\ \alpha_x + i + j & \text{if } x < i \text{ and } \alpha_x > j \\ \beta_{x-i} + j & \text{if } x > i \text{ and } x \leq i + j \\ \alpha_{x-i-j} & \text{if } x > i + j \text{ and } \alpha_{x-i-j} \leq j \\ \alpha_{x-i-j} + i + j & \text{if } x > i + j \text{ and } \alpha_{x-i-j} > j \end{cases}$$

As before, the balloon notation is not associative. Unlike 2413-balloons, which have to be interpreted as right-associative, generalized balloons can use brackets to define associativity. Note that the 2413-balloon defined in Section 2 are written as $2413 \circledcirc_{2,2} \beta$ in our generalized notation.

We remark that for any α and any β, we have $\alpha \circledcirc_{0,0} \beta = \alpha \oplus \beta$, and we can easily determine $\mu[\alpha \oplus \beta]$ using results from Propositions 1 and 2 of Burstein, Jelínek, Jelínková and Steingrímsson [3].

7.2 Generalised 2413-balloons

If we restrict α to 2413, then, up to symmetry, there are seven possible values for the indexes: $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, and $(2,2)$. Theorem 12 handles the case where the indexes are $(2,2)$, and [3] handles the case where the indexes are $(0,0)$. For the other indexes, we have

Conjecture 13. Let $\pi = 2413 \circledcirc_{i,j} \beta$, where $(i,j) \in \{(0,1),(0,2),(1,1),(1,2)\}$.

Then

$$\mu[\pi] = \begin{cases} 0 & \text{If } (i,j) = (0,1) \text{ and } \beta = \tau \oplus 1 \\ 0 & \text{If } (i,j) = (0,2) \text{ and } \beta = \tau \ominus 1 \\ 0 & \text{If } (i,j) = (1,1) \text{ and } \beta = 1 \ominus \tau \text{ or } 12 \\ 0 & \text{If } (i,j) = (1,2) \text{ and } \beta = 1 \oplus \tau \\ \mu[\beta] & \text{Otherwise.} \end{cases}$$

and

Conjecture 14. Let $\pi = 2413 \circledcirc_{1,0} \beta$. Then

$$\mu[\pi] = \begin{cases} 6 & \text{If } \beta = 1 \\ -2 & \text{If } \beta = 21 \\ 0 & \text{If } \beta = 312 \\ 2\mu[\beta] & \text{If } \beta = 2413 \circledcirc_{1,0} \gamma \\ \mu[\beta] & \text{Otherwise.} \end{cases}$$

We remark here that Theorem 12 and Conjecture 14 have a very similar structure. It is not clear to us whether this similarity is coincidental, or whether there is some deeper reason.

7.3 Bounding the Möbius function on hereditary classes

Corollary 24 in Burstein, Jelínek, Jelínková and Steingrímsson [3] gives us that if π is separable, then $\mu[\pi] \in \{0, \pm 1\}$. The simple permutations in the hereditary class of separable permutations are 1, 12, and 21. In Remark 10 we have unbounded growth where the simple permutations in the hereditary class are just 1, 12, 21, 2413, and 25314, so adding 2413 and 25314 to the simple permutations moves us from bounded growth to unbounded growth. This then leads to:

Question 15. If C is a hereditary class containing just the simples 1, 12, 21 and 2413, and $\pi \in C$, then is $\mu[\pi]$ bounded? Further, if D is a hereditary class containing just the simples 1, 12, 21, 2413, and 3142, and $\pi \in D$, then is $\mu[\pi]$ bounded?

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